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A CHARACTERIZATION OF DIRICHLET-INNER FUNCTIONS

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ABSTRACT. We study a concept of inner function suited to Dirichlet-type spaces. We characterize Dirichlet-inner functions as those for which both the space and multiplier norms are equal to 1.

1. INTRODUCTION

1.1. Dirichlet-type and weighted Hardy spaces. The Hardy space H^2 is a well known object in complex function theory consisting of all holomorphic functions f over the unit disc \mathbb{D} of the complex plane for which the Taylor coefficients $\{\hat{f}(k)\}_{k \in \mathbb{N}}$ are in ℓ^2 , with the corresponding norm defined as

$$\|f\|_{H^2} = \left(\sum |\hat{f}(k)|^2 \right)^{1/2}. \quad (1)$$

We refer the reader to [9, 13] for the classical theory of H^2 . H^2 admits an integral representation in terms of the L^2 norm with respect to Lebesgue measure over the unit circle.

Consider a measure μ defined over $\overline{\mathbb{D}}$ and define the *Dirichlet-type space induced by μ* , D_μ , as the space of holomorphic functions f over \mathbb{D} with finite norm

$$\|f\|_{D_\mu} := \left(\|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 d\mu(z) \right)^{1/2}. \quad (2)$$

We denote the inner product associated to this norm as $\langle \cdot, \cdot \rangle_{D_\mu}$.

Important particular cases include, again, the Hardy space (for instance, when μ is any absolutely continuous measure with bounded density and compact support), and the classical *Dirichlet space*, denoted D , where μ is the normalized Lebesgue measure of area A over

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\mathbb{D} . The theory about the Dirichlet space has seen an increased interest in the last few decades. Useful references on Dirichlet-type spaces include [5, 11, 17].

Here we focus on the particular case of a radial weight, that is, the case in which $d\mu(z) = d\mu(|z|)$ for all $z \in \mathbb{D}$, because such spaces also admit a norm in terms of the Taylor coefficients: Let $\omega = \{\omega_k\}_{k \in \mathbb{N}}$ be a sequence of positive real numbers and define the *weighted Hardy space induced by ω* , H_ω^2 , as the space of all holomorphic functions f over \mathbb{D} with Taylor coefficients $\{\hat{f}(k)\}_{k \in \mathbb{N}}$ with finite norm

$$\|f\|_\omega := \left(\sum |\hat{f}(k)|^2 \omega_k \right)^{1/2}. \quad (3)$$

We will normalize $\omega_0 = 1$, and assume that there exists some constant $C > 1$ such that for all $k \in \mathbb{N}$ we have

$$\omega_k \leq Ck^2 \quad (4)$$

and

$$\omega_k \leq \omega_{k+1}. \quad (5)$$

The first property ensures these weighted Hardy spaces are also (radially weighted) Dirichlet-type spaces while the second guarantees that the *shift operator* S (of multiplication by the independent variable z) becomes a bounded operator that increases the norm of every element in the space. We say that a space satisfying (5) has the *expansive shift property*. This excludes the classical Bergman space A^2 , where $\omega_k = 1/(k+1)$ but it is satisfied by H^2 and D . Dirichlet-type spaces with a radial weight and the expansive shift property will be our setting.

1.2. Inner functions and multipliers. A function $f \in H^2$ is called *inner* if it satisfies the following two conditions:

$$|f(z)| \leq 1 \quad \forall z \in \mathbb{D} \quad (6)$$

$$|f(e^{i\theta})| = 1 \quad a.e. \theta \in [0, 2\pi) \quad (7)$$

Inner functions play a major role in the function theory of H^2 , and in particular, in the theory of invariant subspaces for the shift operator. A space closely related with H^2 is the space H^∞ of bounded analytic functions over \mathbb{D} . A function g is in H^∞ if it has finite norm, as defined by

$$\|g\|_\infty := \sup_{z \in \mathbb{D}} |g(z)|. \quad (8)$$

Since the L^2 norm over the circle represents the norm in H^2 , it is easy to see that $\|f\|_{H^2} = 1$ for any inner f . The starting point for the present article is the following observation:

Remark 1.1. In the definition of inner functions given above we could express conditions (6) and (7) as

$$\|f\|_{H^2} = \|f\|_{H^\infty} = 1. \quad (9)$$

Back in the setting of Dirichlet-type spaces D_μ , we say that h is a *multiplier* of D_μ if multiplying by h is a bounded operator M_h acting on the space D_μ . If the norm of the operator is identified with the norm of h as a multiplier $\|h\|_{M_{D_\mu}}$, this defines a Banach algebra called the *space of multipliers* of D_μ . As a relevant example, H^∞ is the space of multipliers of H^2 with the norm (8). Observe that the choice of equivalent norm in a space affects the multiplier norm but it does not affect whether or not certain function is a multiplier.

When the concept of inner functions has been extended to Dirichlet-type (and other) spaces, inner functions have been described in terms of some orthogonality properties of the function. Denote by $\delta_{\cdot, \cdot}$ the Kronecker delta.

Definition 1.2. We say that a function f is D_μ -inner if, for $j \in \mathbb{N}$,

$$\langle z^j f, f \rangle = \delta_{0,j}. \quad (10)$$

It is easy to check that inner functions are exactly H^2 -inner functions. The systematic study of D_μ -inner functions goes back to the work of Richter in [15], who showed that D -inner functions are also connected with invariant subspaces of the shift on D . However, examples of some D_μ -inner functions can already be found in the work of Shapiro and Shields ([18]). They provide examples of inner functions with any prescribed finite zero set on the unit disc. For further background on the relation between generalized inner functions, invariant subspaces, and other topics, see [6].

In 1992, Richter and Sundberg ([16]) proved the following connection between inner functions and multipliers, as part of a stronger result:

Theorem 1.3. *Let μ be a harmonic weight and f , D_μ -inner. Then*

$$\|f\|_{M_{D_\mu}} = 1. \quad (11)$$

Another proof was given by Aleman in [1]. See also [11], Theorem 8.3.9.

2. MAIN RESULT

Our main goal in this paper is to prove a reciprocal to Theorem 1.3. As a byproduct, we obtain a relatively small set of test functions that contain all the information on whether a given function is inner.

Theorem 2.1. *Let $f \in D_\mu = H_\omega^2$. Then f is D_μ -inner whenever*

(a) $\|f\|_{D_\mu} = 1$ and for all $k \in \mathbb{N} \setminus \{0\}$ and all $\lambda \in \mathbb{C}$,

$$\|fg_{k,\lambda}\|_{D_\mu}^2 \leq \omega_k + |\lambda|^2,$$

where $g_{k,\lambda}(z) = z^k + \lambda$,

and this holds true whenever

(b)

$$\|f\|_{D_\mu} = \|f\|_{M_{D_\mu}} = 1.$$

Given Theorem 1.3, this is a characterization of Dirichlet-inner functions.

Proof. Clearly, (b) implies (a), since (a) only requires that

$$\|fg\|_{D_\mu} \leq \|g\|_{D_\mu}$$

holds for a subset of all possible $g \in D_\mu$. Therefore what remains is to show that (a) implies D_μ -inner.

To see this, assume f is not D_μ -inner but $\|f\|_{D_\mu} = 1$ and let $k \in \mathbb{N} \setminus \{0\}$ such that

$$\langle z^k f, f \rangle \neq 0. \quad (12)$$

If we find a value of λ such that

$$\|(z^k + \lambda)f\|_{D_\mu}^2 > \|(z^k + \lambda)\|_{D_\mu}^2 = \omega_k + |\lambda|^2, \quad (13)$$

then (a) is not satisfied and we will be done.

Decompose the right-hand side of (13) in terms of inner products as

$$\|(z^k + \lambda)f\|_{D_\mu}^2 = \|z^k f\|_{D_\mu}^2 + |\lambda|^2 \|f\|_{D_\mu}^2 + 2\operatorname{Re}(\overline{\lambda} \langle z^k f, f \rangle). \quad (14)$$

Since the shift is expansive, we have that

$$\|z^k f\|_{D_\mu}^2 \geq \|f\|_{D_\mu}^2 = 1. \quad (15)$$

Using that $\|f\|_{D_\mu} = 1$ and (12), we may choose

$$\lambda = \frac{\langle z^k f, f \rangle}{|\langle z^k f, f \rangle|^2} \frac{\omega_k}{2}, \quad (16)$$

and this will yield

$$\|(z^k + \lambda)f\|_{D_\mu}^2 \geq 1 + |\lambda|^2 + \omega_k. \quad (17)$$

That is, this choice of λ achieves (13) completing the proof. \square

3. FURTHER REMARKS

- (A) The multipliers of D were characterized by Stegenga in [19] in terms of Carleson measures. See also [5]. It would be interesting to describe properties (a) or (b) in Theorem 2.1 from Carleson measures.
- (B) In spaces without the expansive shift property, Theorem 2.1 fails to hold in general as D_μ -inner does not always imply (b). For instance, take the classical Bergman space $A^2 = H_\omega^2$ for $\omega_k = 1/(k+1)$, which admits a representation as the L^2 integral over the disc. Its space of multipliers is H^∞ (as for H^2). Any nonconstant function bounded by 1 will have Bergman norm strictly less than 1, and hence no nonconstant functions satisfy (b), while there is a plethora of nonconstant A^2 -inner functions (as made clear in any of [4, 14, 10, 6]). However, our proof of (b) \Rightarrow (a) $\Rightarrow D_\mu$ -inner works in any weighted Hardy space with the expansive shift property: we do not make use of the assumption (4). Therefore, it seems natural to ask whether Theorem 1.3 holds true on all weighted Hardy spaces with the expansive shift property. It even makes sense to ask for an analogue to Theorem 2.1 on reproducing kernel Hilbert spaces where the shift increases the norm. Advances on the understanding of the multiplier properties of general spaces inner functions have recently been made in [2] and other references therein. This may yield an additional way to prove the result in this article in a more general environment. The dichotomic situation between spaces with or without the expansive shift property is also present in previous work ([7]). In spaces where the shift is contractive, an analogue proof to that of our theorem will show that contractive divisors are inner, although in the most classical case this is already well known.
- (C) Condition (a) in Theorem 2.1 tells us that we only need to check the multiplication properties on a sequence of unidimensional vector subspaces of D_μ . In fact, it can be shown that it suffices to test on a countable set of functions (for example, by taking λ on $\mathbb{Q} + i\mathbb{Q}$). If moreover, we are performing the test on a function whose Taylor coefficients are real, then λ can be taken to be real.

- (D) From Theorem 2.1 we learn that inner functions are exactly those functions that solve the extremal problem

$$\inf \left\{ \frac{\|f\|_{M_{D_\mu}}}{\|f\|_{D_\mu}} : f \in D_\mu \setminus \{0\} \right\}. \quad (18)$$

It seems conceivable that a similar property is satisfied in other spaces like the Bergman space A^2 . We conclude by proposing a few problems along this line:

Problem 3.1. *Let M be a z -invariant subspace of A^2 containing at least one bounded function. Let g be a function solving the extremal problem*

$$\inf \left\{ \frac{\|f\|_{H^\infty}}{\|f\|_{A^2}} : f \in M \setminus \{0\} \right\}. \quad (19)$$

Is it always true that g is a constant multiple of an A^2 -inner function?

In [8], the authors showed that there exist invariant subspaces of A^2 containing no bounded functions and, therefore, we need the hypothesis on bounded functions. See also [3] and the references therein.

A follow up question is a reciprocal to this:

Problem 3.2. *Does the orthogonal projection of the constant function 1 onto M solve the extremal problem (19)?*

The same questions make sense in any reproducing kernel Hilbert space where inner functions are defined.

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